2.2 Limits Involving Infinity

What you will learn about . . .

- Finite Limits as $x \to \pm\infty$
- Sandwich Theorem Revisited
- Infinite Limits as $x \to a$
- End Behavior Models
- "Seeing" Limits as $x \to \pm\infty$

and why . . .

Limits can be used to describe the behavior of functions for numbers large in absolute value.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say "the limit of f as x approaches infinity" we mean the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)

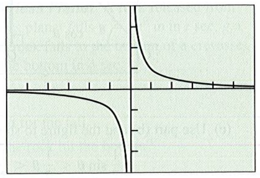
Looking at $f(x) = 1/x$ (Figure 2.9), we observe

(a) as $x \to \infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to\infty} (1/x) = 0,$$

(b) as $x \to -\infty$, $(1/x) \to 0$ and we write

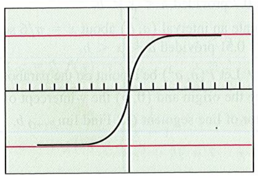
$$\lim_{x\to-\infty} (1/x) = 0.$$



$[-6, 6]$ by $[-4, 4]$

Figure 2.9 The graph of $f(x) = 1/x$.

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.



$[-10, 10]$ by $[-1.5, 1.5]$

(a)

X	Y1
0	0
1	.7071
2	.8944
3	.9487
4	.9701
5	.9806
6	.9864

Y1 $\blacksquare$ X/√ (X² + 1)

X	Y1
-6	-.9864
-5	-.9806
-4	-.9701
-3	-.9487
-2	-.8944
-1	-.7071
0	0

Y1 $\blacksquare$ X/√ (X² + 1)

(b)

Figure 2.10 (a) The graph of $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes, $y = -1$ and $y = 1$. (b) Selected values of f. (Example 1)

DEFINITION Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The graph of $f(x) = 2 + (1/x)$ has the single horizontal asymptote $y = 2$ because

$$\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2 \quad \text{and} \quad \lim_{x\to-\infty}\left(2 + \frac{1}{x}\right) = 2.$$

A function can have more than one horizontal asymptote, as Example 1 demonstrates.

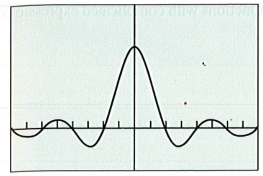
EXAMPLE 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2+1}$.

SOLUTION

Solve Graphically Figure 2.10a shows the graph for $-10 \le x \le 10$. The graph climbs rapidly toward the line $y = 1$ as x moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus $\lim_{x\to\infty} f(x) = 1$. Similarly, as x moves away from the origin to the left, the graph drops rapidly toward the line $y = -1$ and soon appears to overlap the line. Thus $\lim_{x\to-\infty} f(x) = -1$. The horizontal asymptotes are $y = 1$ and $y = -1$.

continued



$[-4\pi, 4\pi]$ by $[-0.5, 1.5]$

(a)

X	Y1
100	-.0051
200	-.0044
300	-.0033
400	-.0021
500	-9E-4
600	7.4E-5
700	7.8E-4

Y1 $\blacksquare$ sin(X)/X

(b)

Figure 2.11 (a) The graph of $f(x) = (\sin x)/x$ oscillates about the x-axis. The amplitude of the oscillations decreases toward zero as $x \to \pm\infty$. (b) A table of values for f that suggests $f(x) \to 0$ as $x \to \infty$. (Example 2)

Confirm Numerically The table in Figure 2.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since f is an odd function of x, we can expect its values to approach -1 in a similar way as $x \to -\infty$.

Now Try Exercise 5.

Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm\infty$.

EXAMPLE 2 Finding a Limit as x Approaches ∞

Find $\lim_{x\to\infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$.

SOLUTION

Solve Graphically and Numerically The graph and table of values in Figure 2.11 suggest that $y = 0$ is the horizontal asymptote of f.

Confirm Analytically We know that $-1 \le \sin x \le 1$. So, for $x > 0$ we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Therefore, by the Sandwich Theorem,

$$0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Since $(\sin x)/x$ is an even function of x, we can also conclude that

$$\lim_{x\to-\infty}\frac{\sin x}{x} = 0.$$

Now Try Exercise 9.

Limits at infinity have properties similar to those of finite limits.

THEOREM 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \ne 0$
6. *Power Rule:* If r and s are integers, $s \ne 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

EXAMPLE 3 Using Theorem 5

Find $\displaystyle\lim_{x\to\infty} \frac{5x + \sin x}{x}$.

SOLUTION

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

So,

$$\begin{aligned}\lim_{x\to\infty} \frac{5x + \sin x}{x} &= \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{\sin x}{x} && \text{Sum Rule}\\ &= 5 + 0 = 5. && \text{Known values}\end{aligned}$$

Now Try Exercise 25.

EXPLORATION 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x \to \infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x\to\infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x \to \infty$. Can we apply the Sum Rule to $\lim_{x\to\infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln (2x)$ and $g(x) = \ln (x + 1)$. Find the limits as $x \to \infty$ of f and g. Can we apply the Difference Rule to $\lim_{x\to\infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x \to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x\to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x \to a$, we say that $\lim_{x\to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 2.9, page 70), we observe that

$$\lim_{x\to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x\to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

DEFINITION Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

EXAMPLE 4 Finding Vertical Asymptotes

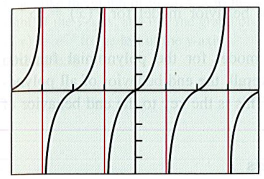
Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

SOLUTION

The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The line $x = 0$ is the only vertical asymptote. ***Now Try Exercise 27.***



$[-2\pi, 2\pi]$ by $[-5, 5]$

Figure 2.12 The graph of $f(x) = \tan x$ has a vertical asymptote at

$\ldots, -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$ (Example 5)

We can also say that $\lim_{x\to 0} (1/x^2) = \infty$. We can make no such statement about $1/x$.

EXAMPLE 5 Finding Vertical Asymptotes

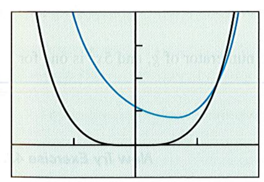
The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x\to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x\to a^-} \tan x = \infty,$$

as suggested by Figure 2.12. ***Now Try Exercise 31.***

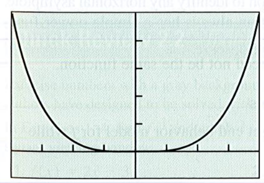
You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x\to 0} (\sin x)/x = 1$.

$y = 3x^4 - 2x^3 + 3x^2 - 5x + 6$



$[-2, 2]$ by $[-5, 20]$

(a)



$[-20, 20]$ by $[-100000, 500000]$

(b)

Figure 2.13 The graphs of f and g, (a) distinct for $|x|$ small, are (b) nearly identical for $|x|$ large. (Example 6)

End Behavior Models

For numerically large values of x, we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

EXAMPLE 6 Modeling Functions For $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that while f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

SOLUTION

Solve Graphically The graphs of f and g (Figure 2.13a), quite different near the origin, are virtually identical on a larger scale (Figure 2.13b).

Confirm Analytically We can test the claim that g models f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4}\\ &= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right)\\ &= 1,\end{aligned}$$

convincing evidence that f and g behave alike for $|x|$ large. ***Now Try Exercise 39.***

DEFINITION End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$.

If one function provides both a left and right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ (b) $g(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

SOLUTION

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is one for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is one for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. *Now Try Exercise 43.*

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

EXAMPLE 8 Finding End Behavior Models

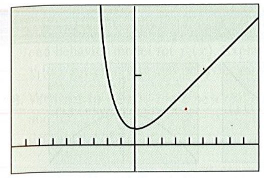
Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

SOLUTION

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty}\left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

continued



[–9, 9] by [–2, 10]

Figure 2.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty}\left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 2.14 supports these end behavior conclusions. *Now Try Exercise 45.*

"Seeing" Limits as $x \to \pm\infty$

We can investigate the graph of $y = f(x)$ as $x \to \pm\infty$ by investigating the graph of $y = f(1/x)$ as $x \to 0$.

EXAMPLE 9 Using Substitution

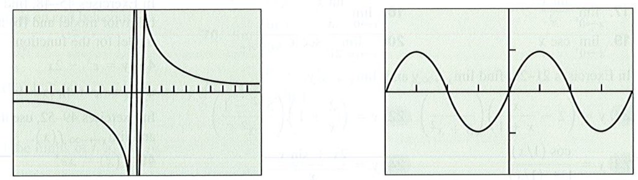
Find $\lim_{x\to\infty} \sin(1/x)$.

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 2.15b), we find

$$\lim_{x\to\infty} \sin 1/x = \lim_{x\to 0^+} \sin x = 0.$$

Now Try Exercise 49.



[–10, 10] by [–1, 1]

(a)

[–2π, 2π] by [–2, 2]

(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

Quick Review 2.2 *(For help, go to Sections 1.2 and 1.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$ **2.** $f(x) = e^x$

3. $f(x) = \tan^{-1} x$ **4.** $f(x) = \cot^{-1} x$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1$, $g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1$, $g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^{-x}$

9. $f(x) = \frac{\ln x}{x}$

10. $f(x) = \left(x + \frac{1}{x}\right)\sin x$

Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$. **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\dfrac{1}{x}\right)$ **2.** $f(x) = \dfrac{\sin 2x}{x}$

3. $f(x) = \dfrac{e^{-x}}{x}$ **4.** $f(x) = \dfrac{3x^3 - x + 1}{x + 3}$

5. $f(x) = \dfrac{3x + 1}{|x| + 2}$ **6.** $f(x) = \dfrac{2x - 1}{|x| - 3}$

7. $f(x) = \dfrac{x}{|x|}$ **8.** $f(x) = \dfrac{|x|}{|x| + 1}$

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. $\displaystyle\lim_{x\to\infty} \frac{1 - \cos x}{x^2}$ **10.** $\displaystyle\lim_{x\to-\infty} \frac{1 - \cos x}{x^2}$

11. $\displaystyle\lim_{x\to-\infty} \frac{\sin x}{x}$ **12.** $\displaystyle\lim_{x\to\infty} \frac{\sin(x^2)}{x}$

In Exercises 13–20, use graphs and tables to find the limits.

13. $\displaystyle\lim_{x\to2^+} \frac{1}{x - 2}$ **14.** $\displaystyle\lim_{x\to2^-} \frac{x}{x - 2}$

15. $\displaystyle\lim_{x\to-3^-} \frac{1}{x + 3}$ **16.** $\displaystyle\lim_{x\to-3^+} \frac{x}{x + 3}$

17. $\displaystyle\lim_{x\to0^+} \frac{\operatorname{int} x}{x}$ **18.** $\displaystyle\lim_{x\to0^-} \frac{\operatorname{int} x}{x}$

19. $\displaystyle\lim_{x\to0^+} \csc x$ **20.** $\displaystyle\lim_{x\to(\pi/2)^+} \sec x$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \dfrac{x}{x + 1}\right)\left(\dfrac{x^2}{5 + x^2}\right)$ **22.** $y = \left(\dfrac{2}{x} + 1\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$

23. $y = \dfrac{\cos(1/x)}{1 + (1/x)}$ **24.** $y = \dfrac{2x + \sin x}{x}$

25. $y = \dfrac{\sin x}{2x^2 + x}$ **26.** $y = \dfrac{x \sin x + 2 \sin x}{2x^2}$

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \dfrac{1}{x^2 - 4}$ **28.** $f(x) = \dfrac{x^2 - 1}{2x + 4}$

29. $f(x) = \dfrac{x^2 - 2x}{x + 1}$ **30.** $f(x) = \dfrac{1 - x}{2x^2 - 5x - 3}$

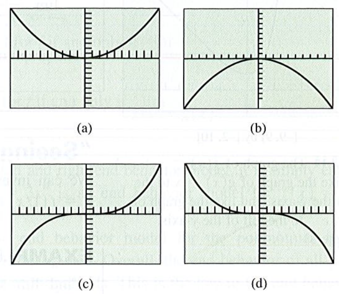
31. $f(x) = \cot x$ **32.** $f(x) = \sec x$

33. $f(x) = \dfrac{\tan x}{\sin x}$ **34.** $f(x) = \dfrac{\cot x}{\cos x}$

In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \dfrac{2x^3 - 3x^2 + 1}{x + 3}$ **36.** $y = \dfrac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$

37. $y = \dfrac{2x^4 - x^3 + x^2 - 1}{2 - x}$ **38.** $y = \dfrac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$



In Exercises 39–44, **(a)** find a power function end behavior model for f. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$ **40.** $f(x) = -4x^3 + x^2 - 2x - 1$

41. $f(x) = \dfrac{x - 2}{2x^2 + 3x - 5}$ **42.** $f(x) = \dfrac{3x^2 - x + 5}{x^2 - 4}$

43. $f(x) = \dfrac{4x^3 - 2x + 1}{x - 2}$ **44.** $f(x) = \dfrac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$ **46.** $y = x^2 + e^{-x}$

47. $y = x + \ln|x|$ **48.** $y = x^2 + \sin x$

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$ **50.** $f(x) = x^2 e^{-x}$

51. $f(x) = \dfrac{\ln|x|}{x}$ **52.** $f(x) = x \sin\dfrac{1}{x}$

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x \to -\infty$, **(b)** $x \to \infty$, **(c)** $x \to 0^-$, and **(d)** $x \to 0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \ge 0 \end{cases}$

54. $f(x) = \begin{cases} \dfrac{x - 2}{x - 1}, & x \le 0 \\ 1/x^2, & x > 0 \end{cases}$

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\displaystyle\lim_{x\to1} f(x) = 2,\ \lim_{x\to5^-} f(x) = \infty,\ \lim_{x\to5^+} f(x) = \infty,$

$\displaystyle\lim_{x\to\infty} f(x) = -1,\ \lim_{x\to-2^+} f(x) = -\infty,$

$\displaystyle\lim_{x\to-2^-} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 0$

56. $\displaystyle\lim_{x\to2} f(x) = -1,\ \lim_{x\to4^+} f(x) = -\infty,\ \lim_{x\to4^-} f(x) = \infty,$

$\displaystyle\lim_{x\to\infty} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 2$

57. Group Activity ***End Behavior Models*** Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. Writing to Learn Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c}(f(x) + g(x))$ be determined? Explain.

Standardized Test Questions

59. True or False It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. True or False If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer.

61. Multiple Choice $\displaystyle\lim_{x\to2^-} \frac{x}{x - 2} =$

(A) $-\infty$ **(B)** ∞ **(C)** 1 **(D)** $-1/2$ **(E)** -1

You may use a graphing calculator to solve the following problems.

62. Multiple Choice $\displaystyle\lim_{x\to0} \frac{\cos(2x)}{x} =$

(A) 1/2 **(B)** 1 **(C)** 2 **(D)** cos 2 **(E)** does not exist

63. Multiple Choice $\displaystyle\lim_{x\to0} \frac{\sin(3x)}{x} =$

(A) 1/3 **(B)** 1 **(C)** 3 **(D)** sin 3 **(E)** does not exist

64. Multiple Choice Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$

(A) x^3 **(B)** $2x^3$ **(C)** $1/x^3$ **(D)** 2 **(E)** 1/2

Exploration

65. *Exploring Properties of Limits* Find the limits of f, g, and fg as $x \to c$.

(a) $f(x) = \dfrac{1}{x},\quad g(x) = x,\quad c = 0$

(b) $f(x) = -\dfrac{2}{x^3},\quad g(x) = 4x^3,\quad c = 0$

(c) $f(x) = \dfrac{3}{x - 2},\quad g(x) = (x - 2)^3,\quad c = 2$

(d) $f(x) = \dfrac{5}{(3 - x)^4},\quad g(x) = (x - 3)^2,\quad c = 3$

(e) Writing to Learn Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c}(f(x) \cdot g(x))$?

Extending the Ideas

66. *The Greatest Integer Function*

(a) Show that

$$\frac{x - 1}{x} < \frac{\operatorname{int} x}{x} \le 1\ (x > 0) \text{ and } \frac{x - 1}{x} > \frac{\operatorname{int} x}{x} \ge 1\ (x < 0).$$

(b) Determine $\displaystyle\lim_{x\to\infty} \frac{\operatorname{int} x}{x}$.

(c) Determine $\displaystyle\lim_{x\to-\infty} \frac{\operatorname{int} x}{x}$.

67. *Sandwich Theorem* Use the Sandwich Theorem to confirm the limit as $x \to \infty$ found in Exercise 3.

68. Writing to Learn Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\displaystyle\lim_{x\to\infty} \frac{\ln x^2}{\ln x}$

70. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\log x}$

71. $\displaystyle\lim_{x\to\infty} \frac{\ln(x + 1)}{\ln x}$

Quick Quiz for AP* Preparation: Sections 2.1 and 2.2

1. Multiple Choice Find $\displaystyle\lim_{x\to3} \frac{x^2 - x - 6}{x - 3}$, if it exists.

(A) -1 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** does not exist

2. Multiple Choice Find $\lim_{x\to2^+} f(x)$, if it exists, where

$$f(x) = \begin{cases} 3x + 1, & x < 2 \\ \dfrac{5}{x + 1}, & x \ge 2 \end{cases}$$

(A) 5/3 **(B)** 13/3 **(C)** 7 **(D)** ∞ **(E)** does not exist

3. Multiple Choice Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}?$$

(A) $y = \frac{3}{2}x$ **(B)** $y = 0$ **(C)** $y = 2/3$ **(D)** $y = 7/5$ **(E)** $y = 3/2$

4. Free Response Let $f(x) = \dfrac{\cos x}{x}$.

(a) Find the domain and range of f.

(b) Is f even, odd, or neither? Justify your answer.

(c) Find $\lim_{x\to\infty} f(x)$.

(d) Use the Sandwich Theorem to justify your answer to part (c).